RAPID METHODS OF APPROXIMATING TO TERMS IN A BINOMIAL EXPANSION

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CONTENTS

1. INTRODUCTION

The labor of computing terms or sums of terms in a binomial expansion can be greatly reduced by the use of certain continuous functions approximating to the values desired. Although these functions are treated in many texts and papers, the simple transformations and substitutions required to obtain formulae directly applicable to computations pertaining to binomial series are in general left for the student to make.

This paper brings together the most suitable of these functions and illustrates their use in order to meet the needs of investigators wishing to obtain numerical results as readily as possible.

2. THE SYMMETRICAL BINOMIAL

The xth term counted from the middle one in the expansion of the symmetrical binomial*

$$\left(\frac{1}{2}+\frac{1}{2}\right)^n=\left(\frac{1}{2}\right)^n+n\left(\frac{1}{2}\right)^n+\frac{n(n-1)}{1\cdot 2}\left(\frac{1}{2}\right)^n+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{2}\right)^n+\ \cdots \tag{1}$$

where the number of the term is

$$r=1,\ 2,\ 3,\ 4\ \ldots \tag{2}$$

is approximately given by the familiar exponential expression (Yule, 1911, p. 300; Merriman and Woodward, 1900, pp. 482–483)

* When n is even, there are an odd number $(n+1)$ of terms, and corresponding to the middle term which is also the maximum, x has the value zero.

When n is odd, there are an even number $(n+1)$ of terms, the two central ones are equal, and the corresponding values of x are, respectively $+\frac{1}{2}$ and $-\frac{1}{2}$.

In both cases

$$r=\frac{n}{2}+1+x$$

or

$$x=r-\frac{n+1}{2}-\frac{1}{2}.$$

$$y=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}=\frac{.399}{\sigma}e^{-\frac{x^2}{2\sigma^2}} \tag{3}$$

where

$$\sigma = \text{the standard deviation} = \tfrac{1}{2}\sqrt{n} \tag{4}$$

and

$$e=2.7183, \text{ the Naperian base of logarithms.}$$

With the aid of a tabulation of this exponential function (Davenport, 1899, p. 21 and 54; Fowle, 1914, pp. 48–54) its value can be quickly found when n and x are given. The difference between this function and the corresponding term of the binomial series decreases as n increases, but the approximation is close even when n is as small as 8, as shown by Table I.

TABLE I

VALUES OF TERMS IN THE BINOMIAL EXPANSION COMPARED WITH THE CORRESPONDING EXPONENTIAL FUNCTION

r	x	y	Terms of the expansion $(\frac{1}{2}+\frac{1}{2})^8$
1	−4	.0052	.0039
2	−3	.0313	.0313
3	−2	.1038	.1092
4	−1	.2198	.2184
5	−0	.2822	.2730

To compute the value of y, substitute 8 for n in the general expression

$$y=\frac{.798}{\sqrt{n}}e^{-\frac{2X^2}{n}} \tag{5}$$

obtained from equations (3) and (4), thus obtaining

$$y=.2822\, e^{-\frac{x^2}{4}}. \tag{6}$$

To find y for a given value of x (x is negative for terms to the left of the middle one), say, $x=-2$, substitute -2 for x in equation (6) and find the value of $e^{-\frac{2^2}{4}}=e^{-1}=\frac{1}{e}$ from table 18 (Fowle, 1914, p. 49). Thus

$$y=(.2822)(.3679)=.1038. \tag{7}$$

To use Davenport's (1899) table (p. 54), find from equation (4)

$$\sigma=\tfrac{1}{2}\sqrt{8}=\sqrt{2}. \tag{8}$$

Thus for $x=-2$

$$\frac{x}{\sigma}=-\frac{2}{\sqrt{2}}=-\sqrt{2}=-1.414 \tag{9}$$

$$\frac{y}{y_0}=e^{-\frac{x^2}{2\sigma^2}}=.3753-.14(.3753-.3246)=.368 \tag{10}$$

and as before

$$y=(.2822)(.368)=.1038. \tag{11}$$

If n equals the odd number, 9, substitution in equation (5) gives

$$y=.266e^{-\frac{2}{9}x^2}, \tag{13}$$

from which the entries in column 3 of Table II were computed.

TABLE II

VALUES OF TERMS IN THE BINOMIAL EXPANSION COMPARED WITH THE CORRESPONDING EXPONENTIAL FUNCTION

r	x	y	Terms of the expansion $(\frac{1}{2}+\frac{1}{2})^9$
1	−4.5	0.0030	0.0020
2	−3.5	0.0175	0.0176
3	−2.5	0.0663	0.0704
4	−1.5	0.1622	0.1640
5	−.5	0.2516	0.2460
6	.5	0.2516	0.2460

A continuous function approximating to the terms of a series is especially useful in computing the sum of a given number of terms, since another continuous function approximating to this sum results from integration. Thus the sum of any number (x) of terms counted from the middle or zero'th term of the binomial expansion is given approximately by the integral

$$\int_o^x ydx=\frac{1}{\sqrt{2\pi}\sigma}\int_o^x e^{-\frac{x^2}{2\sigma^2}}dx=\sqrt{\frac{2}{\pi n}}\int_o^x e^{-\frac{2x^2}{n}}dx \tag{14}$$

where y is given by equation (3), or (5).

Substitution of a new variable $t=x\sqrt{\frac{2}{n}}$ in equation (14) gives the well-known "probability integral"*

$$\int_o^x ydx=\frac{1}{\sqrt{\pi}}\int_o^t e^{-t^2}dt \tag{15}$$

whose values are tabulated in various books on statistics and least squares (for example, Yule, 1911, p. 306; Davenport, 1899, p. 55; Broggi, 1911, pp. 354–356; Fowle, 1914, p. 56; Merriman, 1903, p. 220).

When n is even, if x has the value $\pm\frac{1}{2}$, the integral (14) or (15) gives $\frac{1}{2}$ of the middle term, for $x=\pm(1+\frac{1}{2})$ the integral gives the value of $\frac{1}{2}$ the middle term, plus the next term, and so on.

When n is odd, if x has the value ± 1, the integral (14) or (15) gives one of the two central terms, for $x=\pm 2$ the integral gives one of the two central terms plus that of an adjacent term, and so on.

To find the sum of the first r terms [counting the 1st term as one, see equations (1) and (2)], find the sum from the middle term to, but not including the r th term, and subtract the result from $\frac{1}{2}$, which is the sum of the first $\frac{n+1}{2}$ terms. That is, for odd or even values of n

$$x=\left(r-\frac{n+1}{2}\right) \tag{16}$$

and the sum of first r terms is

$$\frac{1}{2}\left\{1-\frac{2}{\sqrt{\pi}}\int_o^t e^{-t^2}dt\right\} \tag{17}$$

where

$$t=x\sqrt{\frac{2}{n}}=-\frac{n-2r+1}{\sqrt{2n}}. \tag{18}$$

Table III gives the sum of portions of several binomial expansions,

* To determine the probability of obtaining any proportion of an equal number of each of two different things, put

$$\theta=\frac{x}{n} \text{ or } x=n\theta$$

where θ is the difference between the true proportion, $\frac{1}{2}$ and the observed proportion, then

$$t=\theta n\sqrt{\frac{2}{n}}=\theta\sqrt{2n}.$$

This agrees with the result derived by Hatai (1910) from the theorem of Bayes for determining the probability that the true proportion differs by θ from the observed value.

taken from a biological paper (Sumner, 1910, p. 330), and the approximation to these sums given by the "probability integral."

TABLE III

SUMS OF TERMS IN CERTAIN BINOMIAL EXPANSIONS AND CORRESPONDING VALUES OBTAINED FROM THE PROBABILITY INTEGRAL

n	r	$n-2r+1$	$\sqrt{2n}$	t equals $\frac{n-2r+1}{\sqrt{2n}}$	$\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$	$1-\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$	$\frac{1}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt\right]$	Sum of terms of binomial
10	2	7	4.472	1.566	.9732	.0268	.0134	.0193
11	1	10	4.691	2.133	.9974	.0026	.0013	.0005*
11	3	6	4.691	1.280	.9297	.0703	.0351	.0327
11	4	4	4.691	.853	.7725	.2275	.1137	.1111
18	6	7	6.000	1.167	.9011	.0989	.0494	.0588
19	4	12	6.160	1.948	.9941	.0059	.0029	.0022
19	6	8	6.160	1.298	.9336	.0664	.0332	.0318
32	7	19	8.000	2.375	.9992	.0008	.0004	.00041
56	14	29	10.580	2.745	.9999	.0001	.00005	.00005

* The approximation is closer as r increases, and for computing one or two terms the binomial series itself should be used, instead of the integral.

To illustrate the process, determine the sum of the first 7 terms in the expansion

$$\left(\frac{1}{2}+\frac{1}{2}\right)^{32}=\left(\frac{1}{2}\right)^{32}+32\left(\frac{1}{2}\right)^{32}+\frac{32\cdot 31}{1\cdot 2}\left(\frac{1}{2}\right)^{32}+\ \ldots$$

$$r=1,\ 2,\ 3,\ \ldots \tag{19}$$

Here,

$$n=32,\ r=7,\ n-2r+1=19 \tag{20}$$

and $t=\frac{19}{\sqrt{2\cdot 32}}=2.375$ (Table III, line 8). The corresponding value of the integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ (Merriman, 1903, p. 220), is .9992, and $\frac{1}{2}(1-.9992)=.0004$ the approximate value, which agrees well with .00041, the sum of the first 7 terms of the series (equation 19).

To use Table IV (Davenport, 1899, p. 55) the above expression for t must be replaced by

$$\frac{x}{\sigma}=\frac{\frac{n+1}{2}-r}{\frac{1}{2}\sqrt{n}}=\frac{n-2r+1}{\sqrt{n}},$$

and the tabular entry must be subtracted from $\frac{1}{2}$. If $n=32$, and $r=7$,

$$\frac{x}{\sigma}=\frac{32-14+1}{\sqrt{32}}=\frac{19}{\sqrt{32}}=3.360$$

and the corresponding tabular entry is .4996. The approximation to the sum is therefore $.5000-.4996=.0004$.

3. THE ASYMMETRICAL OR SKEW BINOMIAL

A function bearing the same relation to the skew binomial

$$\left.\begin{array}{llll} (p+q)^n=p^n+np^{n-1}q+\frac{n(n-1)}{1\cdot 2}p^{n-2}q^2+\ \ldots \\ r=1, \qquad 2, \qquad\qquad 3, \qquad\qquad \ldots \\ p\neq q \end{array}\right\} \tag{21}$$

as the normal probability function bears to the symmetrical binomial was first found by E. L. Deforest (1882–3). In a review of Deforest's researches, Shinkishi Hatai (1910) presents a derivation of Deforest's (1882–3) formula

$$y=\frac{1}{k\sqrt{2\pi b}}\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax}. \tag{22}$$

Where y is the approximate value of the xth term in the binomial series, the *mean* term $r=(nq+1)$ being the origin,

$$\left.\begin{array}{l} k=1+\frac{1}{12a^2b}+\frac{1}{288(a^2b)^2}+\ \ldots \\ a=\frac{2\mu_2}{\mu_3},\ b=\mu_2 \end{array}\right\} \tag{23}$$

μ_2 and μ_3 are respectively the second and third moments about the mean. These moments* of the binomial series are (Pearson, 1906–7,

* By definition, the second moment $\mu_2=\frac{\Sigma f\cdot x^2}{N}$, where f is the frequency of a deviation x, and $N=\Sigma f=$ the sum of all the frequencies. If the interval between successive terms equals L, instead of unity, as assumed in the preceeding pages,

$$\mu_2=\frac{\Sigma f\cdot x^2\cdot L^2}{N},$$

since the deviation is xL. For a binomial series, the corresponding standard deviation becomes

$$\sigma=\sqrt{\mu_2}=\left\{\sqrt{\frac{\Sigma f\cdot x^2}{N}}\right\}L=\sqrt{npq}L.$$

If the total range, nL is denoted by R, the standard deviation becomes

$$\sigma=\left(\sqrt{npq}\right)\frac{R}{N}=\left(\sqrt{\frac{pq}{n}}\right)R.$$

p. 172; Greenwood, 1913, p. 71)

$$\left.\begin{aligned}\mu_2&=npq\\ \mu_3&=npq(p-q)\end{aligned}\right\}\qquad(24)$$

To illustrate the use of Deforest's rule, compute the rth term of the expansion $(.9+.1)^{20}$. By definition

$$p=.9,\ q=.1,\ n=20,\ 20\times.1+1=3$$

mean term or origin,

$$\mu_2=(20)(.9)(.1)=1.8$$

$$\mu_3=(20)(.9)(.1)(.9-.1)=1.44$$

$$a=\frac{2\times1.8}{1.44}=2.5,\ b=1.8,\ a^2b=11.25$$

$$k=1+\frac{1}{(12)(2.5)^2(1.8)}+\frac{1}{(288)[(2.5)^2(1.8)]^2}=1.0076.$$

Therefore

$$y=.295\left(1+\frac{2}{9}x\right)^{10.25}e^{-2.5x}\qquad(25)$$

where $x=(r-\text{ mean term})=r-3$.
To obtain the third term put $x=0$, then $y=\ .295$.

To obtain the 6th term, put $x=3$, then

$$\begin{aligned}\log .295&=\ \ 9.46982-10\\ 10.25\log\left(\frac{5}{3}\right)=(10.25)(.221858)&=\ \ 2.27404\\ -7.5\log e=(-7.5)(.43429)&=-3.25717\\ \hline \log y&=\ \ 8.48669-10\\ y&=\ \ .03067.\end{aligned}$$

As shown by Hatai (1910), Deforest's formula is equivalent to Pearson's Type III curve whose equation (referred to the mode as origin) is (Pearson 1914, p. xvl)

$$y=y_0e^{-\nu x}\left(1+\frac{\nu x}{m}\right)^m.\qquad(26)$$

In computing terms of a binomial expansion by means of equation (26), the constants have the values (Pearson, 1906–7)

$$m=\frac{4}{npq-\frac{4}{n}}-1,\ \nu=\frac{2}{p-q}$$

$$y_0=\frac{\nu m^m e^{-m}}{\overline{|m+1}}.$$

But a more convenient form

$$y=y_0 e^{-p_1\frac{x}{a_1}}\left(1+\frac{x}{a_1}\right)^{p_1} \tag{27}$$

for numerical work is given by Pearson (1914, pp. xlv and 37). These two forms will be equivalent if $p_1=m$ and $a_1=\frac{m}{\nu}$. Substituting the values of m and ν gives

$$P_1=\frac{4}{\frac{1}{\sigma^2}-\frac{4}{n}}-1=\frac{4\sigma^2(n+1)-n}{n-4\sigma^2}$$

and

$$a_1=\frac{\mathrm{P}_1(\mathrm{p}-\mathrm{q})}{2}$$

where $\sigma^2=npq$. In formula (27) the origin is at the mode, hence $x=r-$mode.

But for Pearson's Type III curve the relation of the mode to mean is (Elderton, 1906, p. 65)

$$\text{Mode}=\text{mean}-\tfrac{1}{2}\frac{\mu_3}{\mu_2}$$

which reduces to

$\text{Mode}=nq+1-\frac{p-q}{2}$ for the binomial series. Therefore in formula (27)

$$x=r-nq-1+\frac{p-q}{2}. \tag{28}$$

Also the simpler expression

$$\frac{1}{\sqrt{2\pi}\,\sigma}=\frac{.399}{\sigma}=\frac{1}{\sqrt{2\pi npq}}$$

can be used for y_0 (Merriman and Woodward, 1900, 482). Table IV gives the terms of the expansion of $(.9+.1)^{20}$ computed from the series, and from Pearson's Type III curve.

TABLE IV

TERMS OF THE EXPANSION $(.9+.1)^{20}$ AND THE CORRESPONDING VALUES OF PEARSON'S TYPE III FUNCTIONS

r	x	$\frac{x}{a_1}=X$	$\log_{10}\frac{(1+X)}{e^x}$	$\log_{10} y$	y	Value of terms, computed from the expansion
1	—1.6	—.390	.046	$\bar{1}.001$	.1002	.122
2	—0.6	—.146	.0052	$\bar{1}.420$	.263	.270
3	0	0	0	$\bar{1}.473$	.297	.285
4	1.4	.342	.020	$\bar{1}.268$	.185	.190
5	2.4	.585	.052	$\bar{2}.939$	.087	.090
6	3.4	.830	.098	$\bar{2}.466$	.029	.032
7	4.4	1.073	.149	$\bar{3}.944$	.009	.009
8	5.4	1.318	.208	$\bar{3}.340$	.002	.002
9	6.4	1.560	.269	$\bar{4}.713$	.0005	.0004
10	7.4	1.805	.336	$\bar{4}.025$	.0002	.0001

To illustrate the use of Pearson's formula (equ. 27) compute y when $r=5$. Then $n=20$, $p=.9$, $q=.1$,

$$\sigma^2=20\times.9\times.1=1.8,\ \sigma=1.342$$

$$4\sigma^2=7.2$$

$$P_1=\frac{7.2\times21-20}{20-7.2}=\frac{131.2}{12.8}=10.25,\ a_1=\frac{10.25(.9-.1)}{2}=4.1$$

and

$$y_0=\frac{.399}{1.342}=.2973 \text{ and } X=\frac{x}{a_1}.$$

To find $\log_{10} y$, subtract P_1 times the value of

$$\left\{\log_{10}(1+X)-X\ \mathrm{Log}_{10}\, e\right\}$$

found in Pearsons' Table XXVI (1914, p. 37), from $\log_{10} y_0$.

For example, if $r=5$, $x=5-3+.4=2.4$, $X=\frac{2.4}{4.1}=.585$

$$\left\{\log_{10}(1+X)-X\ \mathrm{Log}_{10}\, e\right\}=.052$$

$$(10.25)\ (.052)=.534$$

$$\log y_0=\log .2973=\bar{1}.473$$

$$\log y=\bar{2}.939$$

$$y=.087$$

The sum of any number $(r-s)$ of terms measured from the mean is

$$\int_o^x y dx = y_0 \int_o^x e^{-P_1 \frac{x}{a_1}} \left(1 + \frac{x}{a_1}\right)^{P_1} dx \tag{29}$$

where $x = r - s + \frac{p-q}{2}$;

but tables of this integral are not available.

Edgeworth's (1906) theory of skew variation based on an extension of the methods by which the norman law was derived yields much more convenient formulae than Pearson's more general theory. Tolley (1916) concluded that Edgeworth's (1906) approximate rule (see equation (30)) would suffice for climatic data, and has published convenient tables for its application (Tolley 1916, pp. 640–1).

The total area of the generalized curve between the abscissa x_1 and x_2 is

$$A = \int_{x_1}^{x_2} \left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{-\frac{x^2}{2\sigma^2}} dx - \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \left(\frac{k}{1 \cdot 2 \cdot 3}\right)\left(\frac{x^2}{\sigma^2} - 1\right)\right]_{x=x_1}^{x=x_2} \tag{30}$$

(Tolley, 1916, p. 638), where

σ = standard deviation,

x = distance (abscissa) from mean, x_m

μ_3 = third moment about the mean

N = total number of observations,

$k = \frac{\mu_3}{\sigma_3}$.

For the binomial series (equation 21) on page 611

$\sigma = npq$, $\mu_2 = npq\ (p-q)$, $k = \frac{p-q}{\sqrt{npq}}$.

If $y = \frac{x}{\sqrt{2\sigma}}$, equation (30) becomes

$$A = \int_{y_1}^{y_2} \frac{1}{\sqrt{\pi}} e^{-t^2} dt - \left[\frac{1}{\sqrt{2\pi}} e^{-t^2} (2t^2 - 1) \frac{k}{6}\right]_{y=y_1}^{y=y_2}. \tag{31}$$

If y_2 equals ∞, $A = A_r$, area to the right of y_1.

If y_1 equals $-\infty$, $A = A_L$, area to the left of y_2.

That is,

$$A_r=\frac{1}{\sqrt{\pi}}\int_{y_1}^{\infty}e^{-t^2}dt+\left[\frac{1}{\sqrt{2\pi}}e^{-y_1^2}(2y_1^2-1)\frac{k}{6}\right] \tag{32}$$

$$A_L=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{y_2}e^{-t^2}dt-\left[\frac{1}{2\pi}e^{-y_2^2}(2^{y_2^2}-1)\frac{k}{6}\right] \tag{33}$$

but

$$\frac{1}{\sqrt{\pi}}\int_{y_1}^{\infty}e^{-t^2}dt=\tfrac{1}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^{y_1}e^{-t^2}dt\right],\ y_1\overline{\overline{>}}0 \tag{34}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{y_2}e^{-t^2}dt=\tfrac{1}{2}\left[1+\frac{2}{\sqrt{\pi}}\int_0^{y_2}e^{-t^2}dt\right],\ y_2\overline{\overline{>}}0 \tag{35}$$

$$\frac{1}{\sqrt{\pi}}\int_{y_1}^{\infty}e^{-t^2}dt=\tfrac{1}{2}\left[1+\frac{2}{\sqrt{\pi}}\int_0^{y_1}e^{-t^2}dt\right],\ y_1\overline{\overline{<}}0 \tag{36}$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{y_2}e^{-t^2}dt=\tfrac{1}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^{y_2}e^{-t^2}dt\right],\ y_2\overline{\overline{<}}0 \tag{37}$$

and the "probability integrals" in the second members are given in various tables, *e. g.*, Broggi (1911, p. 354), Czuber (1914, p. 437), Fowle (1914, p. 56), Merriman (1903, p. 220) and Pearson (1914, pp. 9–10). The exponential e^{-y^2} has also been tabulated by various authors, *e. g.*, Fowle (1914, p. 48), or may be obtained from a table of the "probability integral" since

$$e^{-y^2}=\frac{\int_{y-\Delta y}^{y+\Delta y}e^{-t^2}dt}{2\Delta y}=\frac{\sqrt{\pi}}{2}\left\{\frac{\frac{2}{\sqrt{\pi}}\int_0^{y+\Delta y}e^{-t^2}dt-\frac{2}{\sqrt{\pi}}\int_0^{y-\Delta y}e^{-t^2}dt}{2\Delta y}\right\}. \tag{38}$$

For example suppose we wish any term or the sum of any number of terms of the series

$$\begin{array}{lllllll} & (.9+.1)^{20}= .122+ & .270+ & .285+ & .190+ & .090+ & .032 \\ r= & \quad 1, & 2, & 3, & 4, & 5, & 6, \\ & & +.009+ & .002+ & .0004+ & .0001, & \\ & & 7, & 8, & 9, & 10. & \end{array} \tag{39}$$

Then

$$p=.9,\ q=.1,\ n=20$$

$$\text{mean}=nq+1=2.0+1=3.0$$

$$\sigma=\sqrt{(20)(.9)(.1)}=1.342$$

$$k=\frac{.9-.1}{1.342}=.596,\ \frac{k}{6}=.0993$$

$y=\dfrac{x}{\sqrt{2}\sigma}=.5268x$, and equations (33), (35), and (37) become

$$A_L=\tfrac{1}{2}\left[1+\frac{2}{\sqrt{\pi}}\int_0^{y_2}e^{-t^2}dt\right]-.0396\ e^{-y_2^2}(2y_2^2-1), \tag{40}$$

for $x \geqq 0$, and

$$A_L=\tfrac{1}{2}\left[1-\frac{2}{\sqrt{\pi}}\int_0^{y_2}e^{-t^2}dt\right]-.0396\ e^{-y_2^2}(2y_2^2-1), \tag{41}$$

for $x<0$, where the absolute values of y are to be substituted.

Let $x=0$, then $y_2=0$ and Equation (40) becomes

$$A_L=\tfrac{1}{2}+.0396=.5396.$$

Let $x=1.5$, then $y_2=.7902$, $y_2^2=.6244$, $e^{-y_2^2}=.5404$ and Equation (40) gives

$$\frac{1.7362}{2}-.0053=.8628.$$

Let $x=-1.5$, then $y_2=-.7902$ and equation (41) gives

$$\frac{1-.7362}{2}-.0053=.1266.$$

The table of values of A_L published by Tolley (1916, pp. 640–641) can be used to facilitate the work, as illustrated by the following computation of the values already obtained.

As before, k has the constant value .596 for all the terms of the series.

Suppose $x=0$, then from the table

	Tabular entry, A_L,	k
	.53989,	.600,
	.52660,	.400,
Diff.	.01329	.200
Diff.	.00027	.004 = .600 − .596
	.53962 = Interpolated value of A_L.	

Suppose $x=1.5$, then $\dfrac{x}{\sigma}=1.117$ and from the table

Tabular entry, A_L value of $\dfrac{x}{\sigma}$

when $k=.6$	.85976	1.10
	.86829	1.15
Diff.	.00853	.05
Diff.	.00289	.017=1.117–1.10

$.86265=A_L$, when $k=6.00$ and $\frac{x}{\sigma}=1.117$

Diff. .00003 correction for reducing to $k=.596$.

.8627 Final interpolated value of A_L

Suppose $x=-1.5$, then $\frac{x}{\sigma}=-1.117$ and from the table

	Tabular entry, A_L	value of $\frac{x}{\sigma}$
when $k=0.600$		
	.13109	-1.10
	.11843	-1.15
Diff.	.01266	.05
Diff.	.0042	$.017=-1.10+1.117$

$.12687=A_L$ when $k=.600$ and $\frac{x}{\sigma}=1.117$

.00003 correction for reducing to $k=.596$

.1269 Final interpolated value of A_L.

To compare these results with the exact values given by the series, we must remember that x is counted from the mean term $np+1=3$, as origin and therefore in general (See equation 21), $r=3+x$.

The first two terms plus $\frac{1}{2}$ the mean term $.122+.270+\frac{.285}{2}=.5345$ corresponds to $x=0$, and agrees well with .5396 obtained from the formula. When $x=-1.5$ the sum is simply the first term, .122. The formula gives .127. When $x=1.5$, the sum is the first 4 terms $.122+.270+.285+.190=.867$. The formula gives .863.

In the case of an extremely asymmetrical binomial in which $\frac{q}{pn}$ is very small, the computation of the general term can be most readily

made by means of Poisson's exponential expansion. (Pearson 1914, p. lxxvii and pp. 113–21.) The smaller the ratio $\frac{q}{pn}$, the more closely the terms of the binomial series

$$(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{1 \cdot 2}p^{n-2}q^2 + \cdots$$

approximate to those of the exponential series

$$e^{-m}\left(1 + m + \frac{m^2}{1 \cdot 2} + \frac{m^3}{1 \cdot 2 \cdot 3} + \cdots\right)$$

$$r = \qquad 1, \quad 2, \quad 3, \qquad\qquad 4$$

where $m = nq$. The expression $e^{-m}\left(\frac{m^x}{x!}\right)$ for the $(x+1)$st or rth term is tabulated for $m = .1$ to 15, and $x = 0$ to 37 by Pearson (1914, pp. 113–21), thus after computing the constant m for a given case, the successive terms of the series can be read off directly from his table. For example, in the expansion $(.9+.1)^{20}$ already considered $m = nq = (20)(.1) = 2$ and the rth term equals $e^{-2}\left(\frac{2^{r-1}}{(r-1)!}\right)$ whose values are entered opposite those of the series in Table V.

TABLE V

TERMS OF THE BINOMIAL SERIES $(.9+.1)^{20}$, AND THE CORRESPONDING VALUES OF POISSON'S EXPONENTIAL EXPANSION

r	$r-1$ equals x	$e^{-2}\left(\frac{2^{r-1}}{\lfloor r-1}\right)$	Terms of $(.9+.1)^{20}$
1	0	.135	.122
2	1	.271	.270
3	2	.271	.285
4	3	.180	.190
5	4	.090	.090
6	5	.036	.032
7	6	.012	.009
8	7	.003	.002
9	8	.0008	.0004
10	9	.0002	.0001

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